Algorithms are useful Understanding them is even better!


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An investigation of students' conceptual understanding of multiplication and division algorithms. A brief overview of research by mathematics educators which provides a selection of student work samples collected as part of research into multiplicative thinking.

This is the first of two articles on the use of a written multiplication algorithm and the mathematics that underpins it. In this first article, we present a brief overview of research by mathematics educators and will then provide a small selection of some of the many student work samples we have collected during our research into multiplicative thinking. We contend that many primary-aged children are taught algorithms for multiplication and division without an appropriate understanding of the mathematical structure and concepts that underpin those algorithms. This is not about demeaning the use of standard algorithms. They have stood the test of time and can be elegant ways of getting a solution. However, imagine the power we give to students if we underpin the strength of algorithms with understanding! In the second article, we elaborate on what we believe are the key mathematical underpinnings of algorithms.

Introduction

Algorithms are very useful methods for calculation when numbers are too large to mentally calculate quickly or accurately. For multiplication, this is generally when there is a need to multiply numbers of two digits or more by another number of a similar magnitude. For example, when attempting to multiply a single-digit number by a double-digit number, students should be considering other strategies, such as applying the distributive property, and exercising their understanding of place value (e.g., 17 × 6 is 10 × 6 which is 60 and 7 × 6 which is 42 so 17 × 6 is 60 + 42 = 102), which allows them to complete these calculations mentally. However, where algorithms are deemed as necessary it would be preferable if the user of the algorithm had an understanding of not only what they were doing, but also, why they are doing it.

An algorithm can be defined as a step-by-step procedure used to solve a problem or complete a task (Anderson et al., 2007). The key here is the word 'procedure' and how this word can sometimes be interpreted. A generally accepted definition of procedure may be a series of actions carried out in a certain order, which seems innocuous enough. However, if the procedure is used without understanding, that is a different matter. Skemp's seminal article differentiated between relational and instrumental learning, with the latter largely equating to "rules without reasons" (1976, p. 20). One of the arguments for the efficacy of algorithms is that they save time and lessen the cognitive load on students, therefore allowing students more 'resources' for problem solving to occur (Merriënboer & Sweller, 2005).

This may be particularly so for students who are cognitively less efficient in mathematics. However, it is important that students do not become too reliant on procedures and algorithms but rather that they have the opportunity to be involved in productive struggle (Jonnson, Norvquist, Liljekvist & Lithner, 2014) to enhance the development of conceptual understanding (Hiebert & Grouws, 2007). It is the exploration of the mathematics behind the procedure that is important, not the uninformed use of the procedure. Whilst we encourage the use of algorithms to aid students in their mathematical development, the use of them without understanding may indeed be impeding that development.

We can choose to teach an algorithm as a purely mechanical way of reaching a solution, but if we do so, much of the potential power of the algorithm is lost. Brosseau (1997) stated that algorithms are designed to be efficient, and to avoid meaning. What he meant by this, was that you can focus on the mechanics without needing to understand what you are doing. For instance, when you "carry the one" you are dealing with it on its face value as being one, not the fact that you are actually renaming 10 ones as one 10, or 10 tens as one hundred etc. Students need to understand and be able to articulate this, which be facilitated by exposure to concrete materials to model the regrouping process. This is supported by Ellis and Yeh (2008), who assert that "traditional algorithms used for multiplication may be efficient but they are not transparent…[that is] they do not allow students to see why they work" (p. 368). Algorithms need to be developed through a thorough understanding of the distributive property, gradually increasing the size of the numbers and developing the grid or area representation for multiplication. Davis (2008), supported this in saying that, "An algorithm for multi-digit whole-number multiplication can be reformatted in a grid, which can connect the standard algorithm to area" (p. 88). Being able to see all the partial products gives the students the opportunity to understand how the multiplication algorithm works. Using a grid representation for multiplication takes some time and effort both on the part of the teachers and the students but this is time well invested. This investment could save a good deal of time later through minimising the need for remediation. This is supported through the work of Englert and Sinicrope (1994) who wrote "although the time spent in developing the multiplication algorithm using this visual approach (grids) is greater than the time needed to use a more traditional approach, less time is needed for review and reteaching. Students are able to attach meaning to the multiplication algorithm" (p. 447).

As noted before, a second article has been written about the written multiplication algorithm. In it, we describe the mathematics that underpins the written algorithm and we outline a teaching sequence and learning progression for developing students' understanding of how and why the written algorithm works.

Evidence

Following is some evidence collected from our research. Semi-structured interviews were conducted with these students. We posed two research questions:

- Are students able to perform the vertical written algorithm for multiplication?
- Are they able to articulate an understanding of the algorithm and why it works?

Sample

The data presented here come from an ongoing study into children's multiplicative thinking. As part of the study, students from Years 5 and 6 were engaged in semi-structured interviews to ascertain their level of understanding of a range of multiplicative concepts, including their strategies for multiplication, one of which was their use of a written method or algorithm. We found that there was great variation across the sample of 81 students, with some demonstrating a strong understanding and efficient use of the written algorithm while others struggled to correctly use and/or articulate about a strategy, including the written algorithm. The samples presented have been purposely selected to show that some students are attempting to use a written algorithm for multiplication without understanding it and/or when the use of a mental strategy might have been more effective. Pseudonyms have been used for the students.

Katie (Year 6)

Katie was asked to calculate the answer for 17 × 6. She set it out as a vertical algorithm (Figure 1) and explained her working in the following way: "Six goes into seven once, write down one and carry one and add it to the 'one' in the 17. Six times two is 12". It was immediately evident that Katie's attempt to use the algorithm was confused.

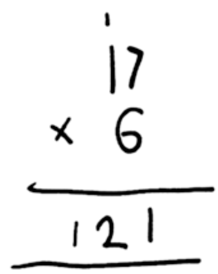


Figure 1.

We wanted to see if different numbers would lead to a different result so we asked Katie to calculate the answer for 13 × 4 (Figure 2). This time she described her working in this way:

"Four times three is 12, write down the two and carry the one. Add it to the one in the 13 to get two. Four times two is eight so the answer is 82".

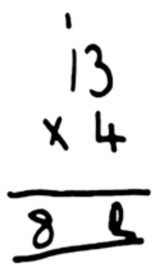


Figure 2.

Katie was asked to check her working for 13 × 4 and this time she arrived at 92! She was then given 20 × 4 and the following conversation ensued:

Katie: I don't need to set that out, I can do it in my head – it's 80.
Int: How did you work it out?
Katie: You do 2 × 4 = 8 and add a zero.

Int: Where did the zero come from?

Katie repeated it and said, “You add the zero back on” but could not say where it came from or what it showed.

Is Katie using algorithms effectively?

In Katie’s articulation of how she solved the 13 × 4 problem there is some mathematics that we can celebrate. For example, she seems to know some of the multiplication facts. This is important, but in itself does not necessarily indicate an understanding of multiplication, particularly if she has merely memorised the number facts. Indeed, there seems to be a lot of mathematical thinking which she does not employ. For instance, the two multiplication examples both involved one case of ‘trading up’ yet Katie’s explanations of each were quite different. She did ‘trade up’ in the second example but combined the tens incorrectly. Also, she made no mention that the one she ‘carried’ was actually worth ten. Saying that, “Add it to the one in the thirteen to get two” does not reflect a strong understanding of place value. Her language in fact hides what is happening. While it may seem like semantics, it actually does not make sense, as adding one to 13 ‘gets’ 14. She really needs to articulate that what she is doing is adding one lot of 10, which can then lead her to “four lots of 10 (4 × 10), add one lot of ten, equals five lots of ten, or 50”. When we talk about the process we actually use partial products so it would seem sensible to underpin this ‘talk’ with work which allows Katie to utilise a model in which the partial products are able to be overtly illustrated. Such a model is the grid (multiplicative array) model for 13 × 4 which can be used to develop the vertical algorithm (Figure 4).

13	4 × 3	=		12
× 4	4 × 10	=	+	40
				52

Figure 3.

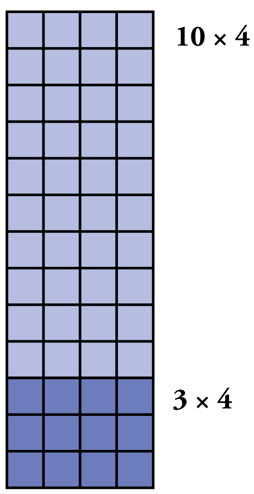


Figure 4.

A question to ask here is: does Katie really need to write a standard algorithm at all? As she has some mastery over multiplication facts, she may actually be better served by using mental computation, by employing the distributive property and perhaps recording a few notes to aid her memory. That is, “four lots of three is 12, four lots of 10 is 40, 40 and 12 is 52”. Of course this is all predicated on Katie understanding that the distributive property can be legitimately used here.

Is Katie able to effectively use algorithms? The evidence suggests not. A standard algorithm is useful but Katie is using it when mental strategies should suffice, and she is using it poorly because she does not understand the parts that make the whole. Katie needs to be assisted to understand how and why the vertical algorithm works, so her classroom experiences should be built around some use of concrete materials such as bundling sticks and MABs, trading games, the distributive property of multiplication, and the use of the array to develop the grid method.

James (Year 6)

James was also asked to work out 17 × 6. Initially he did 6 × 7 = 42, wrote it down and then 1 × 6 = 6 and wrote it in front to get 642 (Figure 5). When asked if he thought the answer was right, James corrected his work. He was then given 34 × 4 and immediately said that the 34 meant 30 + 4. He correctly used the vertical algorithm (Figure 6).

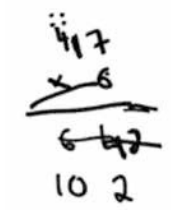


Figure 5.

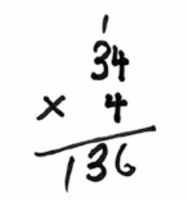


Figure 6.

James was then given a two-digit by two-digit multiplication 29 × 37. He explained it in this way (Figure 7): “Seven times nine is 63, put down the three and carry the six above the two. Three twos make six, and add that to the six I carried—it makes twelve”.

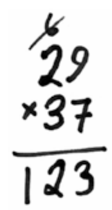


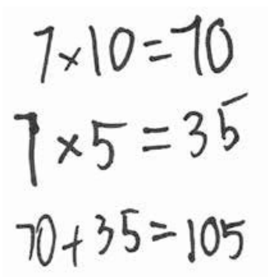
Figure 7.

Is James using algorithms effectively?

In one instance, James was able to use an algorithm to multiply a two-digit number by a one-digit number (34 × 4), but in another instance, his original attempt was incorrect. James showed he is capable of remembering and using some multiplication facts (6 × 7 = 42), and that he was able to partition 34 into 30 and four and to complete the second algorithm correctly. However these did not give him the facility to work with a two-digit by two-digit multiplication problem. In the two-digit by two-digit multiplication James was not prompted to estimate to check the range of his solution before he started the problem, as we wanted to see if he did so of his own accord. He did not estimate before calculating, or to check the reasonableness of the answer (ten lots of 29 is 10 × 29 which equals 290, which is already more than the answer of 123, and 37 lots of 29 is going to be bigger still). He then does not seem to recognise that in carrying out the procedure he has missed some of the partial products. He has not dealt with all of the distributed parts. He has dealt with 7 × 9 = 63 and dealt with 3 × 2 = 6 but has worked with these second set of digits with their face value without any consideration of their place value, that is 30 × 20 = 600. The parts he has totally neglected are 7 × 20 and 30 × 9.

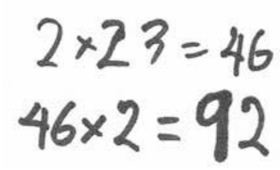
Is James able to effectively use algorithms? The evidence suggests not. Although the fact he recognises partitioning as a viable strategy and manages to calculate the two-digit by one-digit multiplication example, James is still not displaying sound understanding of the structure of multiplication. As with Katie, James would benefit greatly from the use of the array and distributive property to inform the grid representation. This should help James understand the distributed parts he needs to deal with.

Todd (Year 6)



Todd was asked to provide the answer for 7 × 15 and he responded with, “I’d use place value, I’d go 7 times 10 and 7 times 5 and add the results of those together”. The interviewer asked if he would just do that in his head and Todd said that he would. The interview with Todd was conducted at a later date than those with Katie and James, and it was decided to have bundling sticks available to see if they illuminated children’s understanding. When asked if he could use bundling sticks to show 7 × 15, Todd took seven bundles of ten and seven groups of five. Todd also wrote down the working he had done mentally and to show what he had done with the bundling sticks.

Todd was asked to provide the answer for 23 × 4. He used a ‘double-double’ strategy and explained that in his head, he would work out 40 times 2 and 6 times 2 (from the 46 × 2) and add the two answers together. No written algorithm was used. The interviewer asked, “What do you call it when you break up a number like that?” Todd suggested it might be ‘place value’. The interviewer then asked Todd if he had heard of the term ‘partitioning’ and Todd said that he hadn’t.



For 400 × 23, the following discussion occurred.

Todd: Basically it’s the same as 23 × 4 except you’re adding more place value, so you know the answer to that is [wrote 92, circled the two zeros on the 400] and then you add two zeros.

Interviewer: What tells you that you’re ‘adding more place value’?

Todd: The zeros?

Interviewer: What do the two zeros do in the number?

Todd: If there was only one zero, it would be 40, then you could go up to thousands, then forty thousand and so on.

Todd’s work sample follows.

Is Todd using algorithms effectively?

Todd is better positioned to effectively use algorithms than Katie or James as he has some key knowledge upon which algorithms are built. Although Todd does not know the term ‘partitioning’, he does know how to employ it. This situates him well to make the connection between partitioning and the distributive property of multiplication, upon which grid multiplication and the algorithm are founded. Some explicit teaching around this idea, specifically using the terms ‘partitioning’ and the ‘distributive property’, would be beneficial. Todd has some understanding of multiplication when powers of ten are involved. Again, although he said that it is ‘place value’, he is partially correct and would likely benefit from some explicit teaching around the idea of digits moving a place to the left when multiplied by ten. Todd is unlikely to experience difficulties related to the second line of a two-digit algorithm because he seems to be aware that the presence of two zeros in 9200 indicates that he is "adding more place value".

Conclusion

The samples presented indicate some of the typical issues we have seen during our research work. The main concern shown here is a lack of understanding and application of the distributive property when attempting to use a written algorithm. There are two key questions that need to be broached when considering these samples and any other samples of student work. These are:

- What is it about each student's work, which suggests they need to be supported with explicit teaching to enable them to effectively use algorithms?
- What mathematical understandings does each child need to develop which would better equip them to use algorithms?

In order to establish what the issues are, teachers need a deep understanding of the structure of algorithms so that they may identify in very specific terms how to help each student. The key ideas that underpin algorithms are intricately connected and such connections need to be made, explicitly taught to and explored deeply by students.

References

Anderson, J., Briner, A., Irons, C., Shield, M., Sparrow, L., & Steinle, V. (2007). *The Origo handbook of mathematics education.* Origo Education, Brisbane.

Brousseau, G. (1997). *Theory of didactical situations in mathematics.* Dordrecht: Kluwer.

Davis, B. (2008). Is 1 a prime number? Developing teacher knowledge through concept study. *Mathematics Teaching in the Middle School, 14*(2), 86–91.

Ellis, M. & Yeh, C. (2008). Creative arithmetic: Exploring alternative methods. *Teaching Children Mathematics, 14*(6), 367–368.

Englert, G. R. & Sinicrope, R. (1994). Making connections with two-digit multiplication. *Arithmetic Teacher, 41*, 446–48.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In J. Frank, & K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (Vol. 1) (pp. 371–404). Greenwich, CT: Information Age.

Jonsson, B., Norvqvist, M., Liljekvist, Y., & Lithner, J. (2014). Learning mathematics through algorithmic and creative reasoning. *The Journal of Mathematical Behavior, 36*, 20–32.

Merriënboer, J. G., & Sweller, J. (2005). Cognitive load theory and complex learning: Recent developments and future directions. *Educational Psychology Review, 17*(2), 147–177. http://dx.doi.org/10.1007/s10648-005-3951-0

Skemp, R.R. (1976). Relational and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

book review

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Primary Maths Handbook (4th Ed.)

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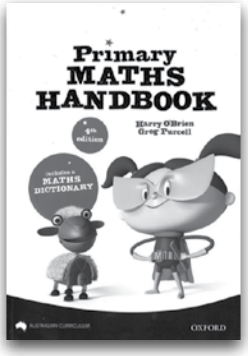

The *Primary Maths Handbook* would make a very useful addition to any primary classroom as a dictionary of common maths terms—an excellent reference book for teachers and parents. Designed to be used by children, with well laid-out graphics and text, it is suitable for Years 3 to 7.

In part one of this fourth edition, key definitions and concepts as used in the *Australian Curriculum: Mathematics*, are organised by mathematical strands:

- number and algebra
- measurement and geometry
- statistics and probability.

The number and algebra section covers number and place value, fractions and decimals, money and financial mathematics, and patterns and algebra. Measurement and geometry looks at using units of measurement, shape, location and transformation, and geometric reasoning. The statistics and probability section describes measuring the element of chance, then outlines the main concepts of data representation and interpretation.

Part two comprises a dictionary of mathematical terms with simple 'in-content' explanations. These are cross-referenced for easy navigation back to part one.

Playful illustrations and clear diagrams add a visual dimension throughout the book, enhancing its appeal to younger readers. This user-friendly resource is well worth purchasing to have a copy in your classroom, the school library, or one for each student.