Multiplicative thinking Much more than knowing multiplication facts and procedures


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Using examples from a current Year 6 research project, this article highlights the importance of a combination of conceptual understanding and procedural fluency in developing multiplicative thinkers.

What characterises multiplicative thinking?

Multiplicative thinking is accepted as a 'big idea' of mathematics (Hurst & Hurrell, 2015; Siemon, Bleckley & Neal, 2012) that underpins important mathematical concepts such as fraction understanding, proportional reasoning, and algebraic thinking. It is characterised by understandings such as the multiplicative relationship between places in the number system, basic and extended number facts, and properties of operations and associated relationships (Hurst & Hurrell, 2015). Siemon, Breed et al. (2006) state that multiplicative thinking is:

- a capacity to work flexibly and efficiently with an extended range of numbers (and the relationships between them);
- an ability to recognise and solve a range of problems involving multiplication and/or division including direct and indirect proportion; and
- the means to communicate this effectively in a variety of ways (e.g., words, diagrams, symbolic expressions, and written algorithms) (p.28)

Multiplicative thinking is a complex set of concepts in which are embedded many connections and relationships. Because of such complexities, this article attempts to consider only a few aspects of multiplicative thinking and how they might be taught. Indeed, we suggest that it is much more than knowing multiplication procedures and number facts and support this with some observations from our recent and on-going research.

The proficiency of understanding

We believe that the development of genuine multiplicative thinking (i.e., more than remembering and recalling number facts) has been hindered through the teaching of procedures at the expense of conceptual understanding. Later we will provide examples of what we mean by this. *The Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority – ACARA , 2015) has adopted four proficiencies (or actions in and with which students should engage), one of which is Understanding. Conceptual understanding is described by Kilpatrick, Swafford, and Findell (2001) as being "… an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods" (p. 118). Kilpatrick et al. (2001) express that conceptual understanding is indicated through a capacity to represent mathematical situations in different ways and is related to the range and richness of the connections made.

ACARA (2015, p. 5) describes the proficiency of Understanding in terms of "a robust knowledge of adaptable and transferable mathematical concepts" and where students "make connections between related concepts" and apply their knowledge to new contexts and situations. In short, it is about links and relationships and knowing how ideas are connected and why processes work as they do. Where the (procedural) Fluency proficiency requires that students choose and use correct procedures in flexible ways, it is only when conceptual understanding is developed that the links and connections between the 'bits' of mathematics actually allow the students to "…see the deeper similarities between superficially unrelated situations" (Kilpatrick, Swafford, & Findell, p. 120) and consequently have less to 'learn'. One way of considering this proposition is to state that learning multiplication facts and the procedure for multiplication is the exercise of Fluency, but multiplicative thinking is the exercise of Understanding.

Conceptual plus procedural

This is not to suggest that learning multiplication facts is not important for the development of multiplicative thinking—the articulation of multiplicative thinking depends to an extent on understanding and remembering multiplication facts. Wong and Evans (2007) make the point that the importance of automaticity of recall can be viewed when it is absent, that is, without automaticity, learning may stall whilst the student tries to search for the required fact. Automaticity enables less working memory to be used on factual recall and more on solving the problem at hand (Willingham, 2009). This gives rise to the slightly challenging notion that although highly desirable, conceptual understanding alone is not sufficient for mathematical proficiency (Bratina & Krudwig, 2003) but rather that a blend of conceptual understanding and procedural fluency is required. However, we assert here that to maximise the effectiveness of procedural fluency, it must be underpinned by conceptual understanding. This is supported by Swan (2007) who argues that whilst automaticity (or rather 'recall') of number facts is highly desirable, it must be based on conceptual understanding of number facts that is built on a robust knowledge of links and connections between them.

Observations from the research—some results and discussion

This article reports on a small part of a current study being conducted by the authors into children's multiplicative thinking. For this part of the study Year 6 students participated in one-on-one interviews to identify aspects of their multiplicative thinking. Interviews were audio recorded and students had access to a range of resources such as counters, bundling sticks and calculators.

Due to the limited scope of this article, we are only able to present findings related to parts of the interview based on the theme of algorithms, the distributive property, and linking to place value.

Algorithms, the distributive property, and linking to place value

The interview questions/tasks that specifically informed this theme are as follows: Can you give an answer for this sum (17 × 6)?

- Children were observed to see if they were able to calculate it mentally or if they needed to use an algorithm.
- If they calculated mentally, they were then prompted with "Please explain how you did it".
- If unable to arrive at an answer, the prompt was "Can you use some of the materials (bundling sticks in sets of ten as well as singles) to help you show what is happening in the sum?"
- If the child had difficulty they were asked if they could demonstrate 12 × 7 for a younger child.
- If further probing was required, the same process was used with 34 × 4.

The 'crunch point'—understanding the algorithm

The interviewees were shown the card 17 × 6 and asked if they could find an answer. Follow-up questions were then asked as shown above. Of the 16 children, all but one chose to use a written vertical algorithm, which in itself is an interesting observation about children in Year 6. Eight children (50% of the sample) could not explain or show the written algorithm in terms of standard place value partitioning, and they were probed further by being asked "Can you use some of these materials to help you show what is happening in the sum?" Bundling sticks in groups of ten and a large number of single sticks were provided. None of the eight children was able to use bundling materials to represent the algorithm. Typical responses from children were to show a group of 17 sticks alongside a group of six sticks (Figure 1). There was no depiction of the desired representation of six groups of 17 sticks (Figure 2) which might reflect an understanding of the standard place value partition and/or the distributive property.

17 × 6

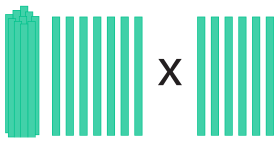


Figure 1. 17 sticks alongside a group of six sticks.

6 × 17

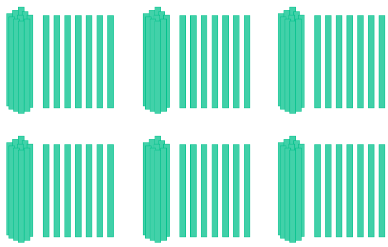
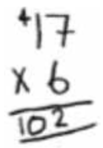


Figure 2. six groups of 17 sticks.

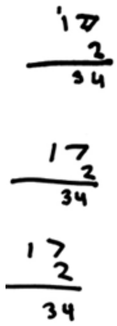
Other probing questions asked of children included "What does the 'four' mean in your sum?"



One student, Angelica, said, "That was the 7 × 6 = 42", and when asked why the four was written at the top, she said, "Add it to the six, the six times one". It is interesting (but perhaps not surprising) that she said 'six times one' rather than 'six times ten'. In contrast, Tilly, who correctly used the algorithm, demonstrated a sound understanding of the procedure by saying, "That's forty that I carried from the first column and that I added to the sixty". Similarly, Christie described her method as "I did 7 × 6 = 42 and then 1 × 6, which is actually 60". She was also able to relate it to the bundling sticks by saying "Take six groups of ten and six groups of seven, and make the 42 into four tens and two ones". Angelica's comment along with the vignettes that follow seem to demonstrate the concern expressed by Young-Loveridge & Mills (2005, p. 641) that "An emphasis on procedural knowledge and rules (without understanding), as reflected in the use of algorithmic approaches to multiplication, may undermine conceptual understanding". Students Tilly and Christie certainly appear to have developed a measure of conceptual understanding while student Angelica and most of the others whose vignettes follow have not appeared to have done so.

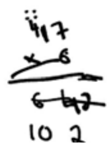
Student Rhiannon

Rhiannon set it out as three separate calculations of 17 × 2 and added the three answers of 34 to get 106 (incorrect). When asked if it was right she did not identify the '6' as being wrong but changed each of the '3's to make 24 and added them to get 76. She explained her reason for doing it as "I find the two times tables very easy—you just have to double the number, I did 17 × 2 three times and added it all up". She seems to have learned a procedure for doubling and has stuck to that as the method for doing calculations whenever she can.



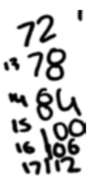
When probed further with the 34 × 4 example and asked if she could show how to do it, she said, "In the right way or the wrong [way]"? She said that knowing that the 3 represented 30 and that it was in tens column would help her do the sum "very much". First she did it in one line and incorrectly arrived at 34 × 4 = 128 and explained that if you did 3 × 4 = 12 and 4 × 4 =16 and added the two to get 28 you would be wrong. She then did it as 34 × 2 twice and added the two result as a vertical addition algorithm (carried the 1) 68 + 68 = 136. She used the same doubling procedure as for the 17 × 6 example.

Student Jane



Jane also used an algorithmic approach. Initially she mentally calculated 6 × 7 = 42 and wrote this down (see left). She then calculated 1 × 6 = 6 and wrote the digit 6 in front of the 42 to make the number 642. This suggests some confusion about partitioning the number 17 and she may be treating the digits 1 and 7 as single digit numbers. Her understanding could hardly be described as robust.

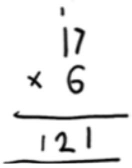
Student Jacinta



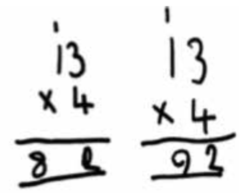
Jacinta first mentally calculated 12 × 6 to get 72, then added another 6 each time, finger counting at each stage. She went from 72 to 78 and recorded 13 showing it to be the 13th multiple of 6, then 78 to 84 (14th multiple) then to 100 not recognising her error. When asked why she did it that way, "That works best for me". She was then asked, "What have you seen other people do?" Her response was, "I don't really like to look at other people's work". This raises some interesting observations. First, she was easily able to recall the number fact 12 × 6 = 72 but had to count on from there. Second, her final comment suggests that there may not be much discussion and sharing of strategies in her mathematics classroom.

Student Letitia

Letitia used a vertical algorithm and arrived at an answer of 121. Her method, as described by her

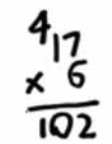


was "6 goes into 7 once, write down 1 and carry 1 and add it to the 1 in the 17. Six times two is twelve so the answer is 121." To probe her understanding, she was given the example 13 × 4. This time, her method was different. She said "4 times 3 is 12, write down the 2 and carry 1". She then added it to the 1 in the 13 to get 2 and said "4 × 2 is 8". Answer = 82.



She was asked to try the same example again (13 × 4). This time she did the same working and arrived at an answer of 92. These responses illustrate considerable confusion in the student's understanding of the mathematics involved in the algorithm. It suggests that she is working with algorithms without the necessary conceptual understanding.

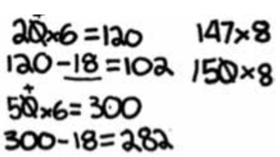
Student Lenny



When asked if he could do 17 × 6, his first reaction was "Not off the top of my head". He used a vertical algorithm and described it procedurally. He was unsure of number facts for 6 × 7; "Is it 48, no 49". He was quite confused when explaining what the 1 in the 17 was worth: "What do you mean?" [then] "It's worth 5, no it's worth 1". Asked about the carried four, he said "That came from the 7 × 6 part; it's worth four". The question was repeated and he still said four ones.

Lenny was shown the bundling sticks including the bundles of ten. "You can't use sticks because it's not a multiple of ten...sorry, a factor of ten". He said he couldn't use the sticks to show the sum. He was asked if he could to do it for 12 × 7 (to show a younger child) but said he couldn't.

Student Holly



Out of the sixteen students who were interviewed, Holly was unique in this choice of method. For 17 × 6 she first worked out 20 × 6 = 120 then took away 18 which is 6 × 3 to get 102. She talked about 'crossing out the zero and then doing the sum and adding back the zero'. When asked "What happens when you put a zero onto a number? What does it do to it?" she said "It makes it part of the ten times table or the five times table". She was probed further; "When you put a zero on how much bigger does it get?" She said "Ten times bigger . . . then if you added another zero it would be 100 times".

In order to further probe her understanding, she was given 47 × 6. "I round it to the nearest number (50) that would bring it to 50 × 6. Five times 6 is 30, add the zero and equals 300. It is 47 so you go 300 – 18 = 282".

For further probing, she was given 147 × 8. She tried to do this in the same way and became 'lost' as the numbers she had to round were too big. She persevered but concluded "Oh wow, this is a harder one". It was suggested that she try the algorithm and she did it correctly and explained the procedure very clearly. However, when asked what the carried numbers were worth, she said they were units.

Linking to place value

The sixteen students who were interviewed seemed to have a robust recall of multiplication facts, and where initial errors may have been made these were self-corrected. Further, many were later able to apply these multiplication facts in the construction and solution of a two-by one-digit multiplication vertical algorithm. However, as is seen from the above vignettes, this facility to recall multiplication facts was not an indicator that the students had a conceptual understanding of multiplication, and could therefore be considered to be multiplicative thinkers.

It may seem that this misunderstanding is more indicative of a lack of place value knowledge but as Major (2012) states, there is a complex multiplicative relationship embedded in place value. Thompson (2009) asserts that place value development runs through three phases: unitary value understanding, quantity value understanding, and then column value understanding. This third understanding, column value understanding, is an important pre-requisite for multiplicative thinking (Thomas, 2004). It is apparent that many of the students whose thinking is described in the vignettes lack this column value understanding and appear unable to determine if their answers obtained through the use of the algorithm are correct or otherwise. However, it is clear that some of the students in the sample do have a measure of conceptual understanding and are able to articulate the correct value of a digit in terms of its column value.

There is a circular piece of reasoning to be followed at this point as, not only does column value understanding support multiplicative thinking, multiplicative understanding supports column value understanding. Further, there is an argument by Graveiimeijer and van Galen (2003) to suggest that a combination of remembering basic multiplication facts and a conceptual understanding of multiplication are both required to move the students through quantity value into column value understanding. The vignettes provided in this article contain evidence that the mastery of the procedural elements of multiplication does not guarantee that multiplicative thinking will be fully developed.

Conclusions

It is apparent from the observations described here that the students in this study have a degree of mathematical fluency with multiplication, but that at least half of them do not have strong conceptual understanding of the same. Please let us state again, that we support the need for this fluency and that knowing multiplication facts is of great value. Yet the evidence collected from the students shows that knowing these facts alone does not provide them a suitable base for a whole range of further mathematical understandings. Just knowing multiplication facts does not make a student a multiplicative thinker. However, some of the students showed that it is entirely possible to develop a deeper understanding of the multiplicative situation. These students were conversant with multiplication facts and also displayed a good level of conceptual understanding of the multiplication algorithm. Further, they were able to illustrate their understanding using concrete materials. Some students showed an awareness of alternative methods of computation (from the algorithm) but their understanding could hardly be described as being robust. Significantly, it is likely that at least some of the students involved in the interviews have been introduced to the written multiplication algorithm without the underpinning conceptual understanding based on place value.

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